Degree of Satisfiability

in the Intuitionistic Propositional Calculus

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Game plan

I took a hint from Bishop: concrete mathematical activity first, philosophy later if I have time (but come talk to me about philosophy).

1. Introduction
2. One-variable equations
3. One-variable: infinite case
4. Multiple variables
5. Philosophy

Groups I

A well-known result from Algebra 1:

Theorem

If $x^2 = 1$ holds for every element of a group G, then G is Abelian (i.e. $xy = yx$ for all $x, y \in G$).

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Take a finite group G. Then either

- *the number of solutions to the equations $x^2 = 1$ in G is less than or equal to $\frac{3}{4}|G|$; or*
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Groups II

Another well-known example:

Theorem ([Gustafson, 1973])

Take a finite group G. Then either

- *the number of pairs (x, y) that solve the equation $xy = yx$ in G is less than or equal to $\frac{5}{8}|G|^2$; or*
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The probability that random x, y fail to satisfy $xy = yx$ is very high (at least $\frac{3}{8}$). "Deceptive" groups, which only barely fail to be Abelian, do not exist.

The general setting

Definition

Take a first-order language $\mathcal{L}$, a finite $\mathcal{L}$-structure M, and an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ in n free variables. We call the quantity

$$\frac{|\{(a_1, \ldots, a_n) \in M^n \mid \varphi(a_1, \ldots, a_n)\}|}{|M|^n}$$

the *degree of satisfiability* of the formula φ in the structure M, and denote it $\mathrm{ds}_M(\varphi)$.

If we can find a constant $\varepsilon > 0$ such that for every finite model M of the theory T, we have either

1. $\mathrm{ds}_M(\varphi) = 1$; or else
2. $\mathrm{ds}_M(\varphi) \leq 1 - \varepsilon$,

then we say that the formula φ has *finite satisfiability gap* ε in T.

Results in groups

Many group-theoretic (and other) equations display this phenomenon:

Theorem

The equation $x^2 = 1$ *has finite satisfiability gap* $\frac{1}{4}$ *(easy). The equation* $x^3 = 1$ *has finite satisfiability gap* $\frac{2}{9}$ *(not so easy, [Laffey, 1976]). Nobody knows if* $x^5 = 1$ *has finite satisfiability gap, and it's not* $\frac{4}{25}$ *(in fact, less than* $\frac{1}{25}$*, due to Terry Wall).*

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Theorem (K. 2019)

The equations $xy^2 = y^2x$, $xy^3 = y^3x$, $xy^{-1} = yx$ and an infinite family of other similar commutator equations in powers $-1, \ldots, 3$ have finite satisfiability gaps - we know tight bounds and can construct groups in which they are taken.

See also [Lescot, 1995] and the survey of [Mann, 2018].

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Might look "toy" at first, but structural information can be obtained from these numbers. One example:

Theorem ([Barry et al., 2006])

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Theorem ([Guralnick and Robinson, 2006])

The only non-solvable groups with $\mathrm{ds}_G(xy = yx)$ *larger than* 0.075 *are* $A_5 \times H$ *where* H *is Abelian.*

Other applications

- Applications to counting algebraic operations (e.g. in a set with two associative operations $\times, +$, the equation $a \times b = a + b$ has finite satisfiability gap).
- Immediate applications to property testing and black-box algebra – e.g. $O(n^2)$ associativity testing.
- *-semiring results: applications to testing regex engines.
- Closely related to isocliny in groups and rings ([Dutta et al., 2017]).

Degree of satisfiability in HAs

Heyting algebras

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Heyting algebras have an equational presentation:

- $x \to x = \top$;
- $x \wedge (x \to y) = x \wedge y$;
- $y \wedge (x \to y) = y$;
- $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
- $+$ bounded lattice axioms $(\bot, \top, \wedge, \vee)$.

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A formula φ of intuitionistic propositional logic is a tautology precisely if the equation $\varphi = \top$ is an identity in every Heyting algebra H.

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Examples:

- $x \vee \neg x = \top$ (*excluded middle*),
- $\neg\neg x = x$ (*double-negation elimination*),
- $(x \to y) \to x = x$ (*Peirce's law*),
- $\neg y \to \neg x = x \to y$ (*contrapositive principle*),
- $(\neg x \to y) \to (x \to y) \to y = \top$ (*LEM - eliminator form*),
- $x \vee y = \neg(\neg x \wedge \neg y)$ (*De Morgan duality*),
- $x \to y = \neg x \vee y$ (*material implication*).

Main Question

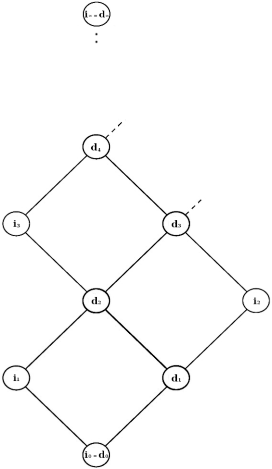
How well can classical principles hold in non-Boolean HAs?
Which equations have finite satisfiability gap?

One-variable equations

The one-variable case

- Joint work with Ben Bumpus (University of Florida, Gainesville). ([Bumpus, K., 2022])
- We classify all one-variable equations w.r.t. finite satisfiability gaps (not just classical principles).
- Possible because the free HA on one generator is well-understood (Rieger, Nishimura).

The Rieger-Nishimura Lattice



$\mathbf{d_0} = \mathbf{i_0} = \bot$

$\mathbf{d_1} = x$

$\mathbf{d_{n+1}} = \mathbf{i_n} \vee \mathbf{d_n}$

$\mathbf{i_\infty} = \mathbf{d_\infty} = \top$

$\mathbf{i_1} = \neg x$

$\mathbf{i_{n+1}} = \mathbf{i_n} \to \mathbf{d_n}$

Classification

Theorem ([Rieger, 1952])

In a Heyting algebra, every system of equations in one free variable x is logically equivalent to an equation of the form $\mathbf{d_n} = \top$ or $\mathbf{i_n} = \top$, where $n \in \mathbb{N}_\infty$.

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So it's enough to classify each $\mathbf{d_n}$ and $\mathbf{i_n}$ with respect to finite gap.

Theorem (Bumpus and K., 2022)

The following equations have finite satisfiability gap:

1. $x = \top$ $(\mathbf{d_1})$,
2. $\neg x = \top$ $(\mathbf{i_1})$,
3. $x \vee \neg x = \top$ $(\mathbf{d_2})$ – *excluded middle.*

No other one-variable equation has finite satisfiability gap. In particular, $\neg\neg x \to x = \top$ $(\mathbf{i_3})$ *doesn't.*

Specific cases

Theorem (Bumpus and K., 2022)

In every finite Heyting algebra H, we have either

1. $\mathrm{ds}_H(x \vee \neg x = 1) = 1$; *(and then H is Boolean) or else*
2. $\mathrm{ds}_H(x \vee \neg x = 1) \leq \frac{2}{3}$

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Takeaway: LEM cannot fail "deceptively", just barely. Contrast with DNE:

Theorem (Bumpus and K., 2022)

For each $n \in \mathbb{N}$, there is a Heyting algebra H where

$$\mathrm{ds}_H(\neg\neg x = x) > 1 - \frac{1}{n}.$$

A word on the proofs

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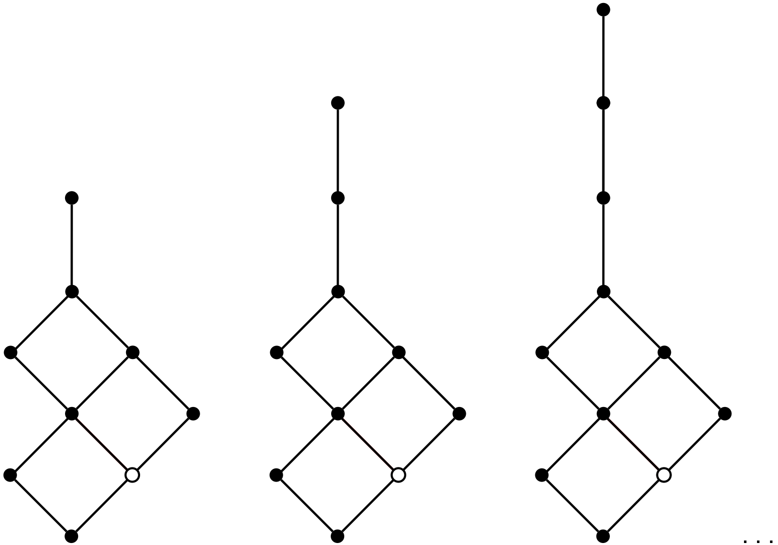
- For the other equations, we construct families of Heyting algebras. Special cases for $\mathbf{i}_2, \mathbf{i}_3$, a general method for the rest.



A family of Heyting algebras showing that $\mathbf{d_5} = \top$ has no finite gap.
The element x not satisfying $\mathbf{d_5} = \top$ is marked in white.

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- E.g. you can define a notion of conjunction and disjunction in commutative BCK-algebras.
- Matt Evans (Oberlin) showed, among other results:

Theorem ([Evans, 2022])

In a bounded commutative BCK-algebra, $x \vee \neg x = \top$ has finite satisfiability gap $\frac{1}{3}$. But $\neg\neg x = x$ does not.

Proof.

First part: follows from the structure theorem for commutative BCK-algebras. Second part: direct construction. □

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Definition

If for every infinite Heyting algebra H, we have either

1. $\{(a_1, \ldots, a_n) \in M^n \mid \neg\varphi(a_1, \ldots, a_n)\} = \emptyset$; or else
2. an injective map $\{(a_1, \ldots, a_n) \in M^n \mid \varphi(a_1, \ldots, a_n)\} \hookrightarrow \{(a_1, \ldots, a_n) \in M^n \mid \neg\varphi(a_1, \ldots, a_n)\}$

then we say that the formula φ has *infinite satisfiability gap*.

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Using topological semantics, we get the following

Corollary (Bumpus, K., 2022)

An infinite T_0 topological space is either discrete or has more non-closed open sets than clopen sets.

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We can use the 1-variable classifiation as a stepping stone, and say something about classical principles.

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Theorem

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Pitts quantifier theory to the rescue

- A celebrated result of ([Pitts, 1992]) lets us "internalize" second-order propositional quantification in the regular intuitionisitic propositional calculus.
- We let $\mathbb{\forall}$ denote Pitts' universal quantifier. $\mathbb{\forall} x.x \vee (x \to y)$ is the best one-variable approximation of the 2nd order formula $\forall x.x \vee (x \to y)$.
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The analogue FAILS if you replace the formula $\forall$ with the equation $(\mathbb{\forall} y.\varphi(x, y)) = \top$.

Fails, but works

We can, however, fix the theorem in a special case:

Theorem (Bumpus and K., 2022)

Take an equation $\varphi(x, y) = \top$ where $\varphi(x, y)$ is a **classical principle**. *If $\varphi(x, y) = \top$ has finite satisfiability gap in T, then so does the equation $\forall y.\varphi(x, y) = \top$.*

This result actually holds.

Pitts test

By computing $\forall x.\varphi(x, y)$, the following have no satisfiability gap:

1. $(x \to y) \to x = x$ (*Peirce's law*),
2. $\neg y \to \neg x = x \to y$ (*contrapositive principle*),
3. $(\neg x \to y) \to (x \to y) \to y = \top$ (*LEM - eliminator form*),
4. $x \vee (x \to y) = \top$ (*LEM - bottomless*).

It settles about 90% of questions you might have. But inconclusive on:

1. $x \vee y = \neg(\neg x \wedge \neg y)$ (*De Morgan duality*),
2. $x \to y = \neg x \vee y$ (*material implication*)

where the Pitts reduct is $x \vee \neg x = \top$. The former has gap $\frac{1}{3}$ (K. 2023), the latter has no gap (but the proof is difficult (Bumpus and K., 2022).

Philosophy

Logical anti-exceptionalism

A philosophical position associated mainly with Quine, Maddy, Hjortland, Williamson. Short version:

- Logic: just another science, revised empirically.
- Not analytic, not *a priori*.

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[Hjortland, 2019]'s summary of Williamson's account has worked-out examples.

Anti-exceptionalist desideratum

(My formulation, but one that mostly follows [Williamson, 2013]): sometimes, we can observe/verify/falsify propositions via logic-independent (e.g. physical) means.

Assume that

1. $A_1, \ldots, A_n, B$ are all independently verifiable/falsifiable;
2. $A_1, \ldots, A_n$ are independently verified; and
3. B follows from $A_1, \ldots, A_n$ according to our logic $\mathfrak{L}$.

If we can then independently falsify B, that counts as scientific evidence against our logic $\mathfrak{L}$.

Application

If we're unsure whether classical or intuitionistic logic applies to observational phenomena, we can apply the degree of satisfiability results for LEM. Whenever the classical laws of propositional logic fail in a given setting, the evidence disconfirming them should be abundant. In fact:

Theorem (Bumpus and K., 2022)

For any natural $n \geq 2$, let f_n be one of the formulae $\{i_{n+1}, d_n\}$ in the Rieger-Nishimura lattice. There is a strictly monotone function g such that, given any Heyting algebra H, if $\mathrm{ds}_H(d_2) > \frac{2}{g(n)}$, then $\mathrm{ds}_H(f_n) = 1$.

So if some observational phenomenon is governed by *pure intuitionistic logic*, then LEM fails for almost every proposition.

Application

If we're unsure whether classical or intuitionistic logic applies to observational phenomena, we can apply the degree of satisfiability results for LEM. Whenever the classical laws of propositional logic fail in a given setting, the evidence disconfirming them should be abundant. In fact:

Theorem (Bumpus and K., 2022)

For any natural $n \geq 2$, let f_n be one of the formulae $\{i_{n+1}, d_n\}$ in the Rieger-Nishimura lattice. There is a strictly monotone function g such that, given any Heyting algebra H, if $\mathrm{ds}_H(d_2) > \frac{2}{g(n)}$, then $\mathrm{ds}_H(f_n) = 1$.

So if some observational phenomenon is governed by *pure intuitionistic logic*, then LEM fails for almost every proposition.
Hope: similar results could lead to an argument for the existential soundness of classical reasoning (but don't hold your breath).

Thank you!
Questions?

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X: LEM Proof

Proof.

Take any maximal non-central element $\sigma \in \mathbf{S}$. The map $f : \mathbf{Z}(H) \to \mathbf{S}$ given by the equation

$$f(x) = \begin{cases} \sigma \wedge x & \text{if } \sigma \vee x = \top \\ \sigma \wedge \neg x & \text{otherwise} \end{cases}$$

is well-defined and two-to-one. If $a \vee b$ and $a \wedge b$ both belong to $\mathbf{Z}(H)$, then so do a and b, which guarantees well-definedness. Injectivity: take two central elements c, d such that $\sigma \vee c = \top$ and $\sigma \vee d = \top$. Assume that $f(c) = \sigma \wedge c = \sigma \wedge d = f(d)$. Then we have that

$$c = c \vee (\sigma \wedge c) = c \vee (\sigma \wedge d) = (c \vee \sigma) \wedge (c \vee d) = \top \wedge (c \vee d) = c \vee d.$$

By a similar argument, we have $d = c \vee d$. Thus $c = c \vee d = d$. □

X: Structural Information

We do get some:

Proof.

Let H be a finite Heyting algebra. If $\mathrm{ds}_H(x \vee \neg x = \top) \geq 1/2$, then $\mathbf{Z}(H) = H_{\neg\neg}$. Since $\mathbf{Z}(H)$ and $H_{\neg\neg}$ are both Boolean algebras, their sizes are powers of two. Since $\mathbf{Z}(H) \subseteq H_{\neg\neg}$, this means that either $\mathbf{Z}(H) = H_{\neg\neg}$ or $H_{\neg\neg} = H$. In the latter case double-negation elimination holds everywhere. But then H is a Boolean algebra which implies that the law of excluded middle holds everywhere as well. □

We note that the $\frac{1}{2}$ bound obtained above is not tight. A tight bound ($\frac{2}{5}$) follows immediately from other results.

X: Pitts Quantifier Theorem

Theorem ([Pitts, 1992])

Take a finite sequence of propositional variables $\overline{x}$, and a propositional variable y not contained in $\overline{x}$. Let $\Phi(\overline{x}, y)$ denote a formula of intuitionistic propositional calculus containing only the variables in $\overline{x}, y$. Then we can find a propositional formula $\mathbb{V}y.\Phi(\overline{x}, y)$ so that the following all hold:

1. *The formula $\mathbb{V}y.\Phi(\overline{x}, y)$ contains only the variables in $\overline{x}$.*
2. *For any propositional formula $\Psi(\overline{x})$, intuitionistic logic proves the implication $\Psi(\overline{x}) \to \mathbb{V}y.\Phi(\overline{x}, y)$ precisely if it proves $\Psi(\overline{x}) \to \Phi(\overline{x}, y)$.*
3. *Given any propositional formula Ψ, intuitionistic logic proves all implications $\mathbb{V}y.\Phi(\overline{x}, y) \to \Phi(\overline{x}, \Psi)$, where $\Phi(\overline{x}, \Psi)$ denotes the formula obtained by substituting the formula Ψ for the propositional variable y everywhere in Φ.*

X: Rieger-Nishimura Lattice

Definition

We define the sequences *d* and *i* of *disjunctive* and *implicative* *Rieger-Nishimura terms* by mutual recursion as follows:

$$\begin{aligned} \mathbf{d_0} &= \bot, & \mathbf{i_0} &= \bot, \\ \mathbf{d_1} &= y, & \mathbf{i_1} &= \neg y, \\ \mathbf{d_{n+1}} &= \mathbf{i_n} \vee \mathbf{d_n}, & \mathbf{i_{n+1}} &= \mathbf{i_n} \to \mathbf{d_n}, \\ \mathbf{d_\infty} &= \top & \mathbf{i_\infty} &= \top. \end{aligned}$$

The *Rieger-Nishimura lattice* consists of the terms $\mathbf{d_n}, \mathbf{i_n}$ for all $n \in \mathbb{N} \cup \{\infty\}$ in the free variable y, equipped with the logical ordering.

[Kocsis, 2020] [Bumpus and Kocsis, 2022]